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LETTER TO THE EDITOR

On the propagator related to an electron in a random potential

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Abstract. We provide physical and mathematical reasons by which propagators associated with non-local actions may not satisfy the composition property and may not be of the Van Vleck-Pauli formula either. Furthermore, we demonstrate that the Feynman and Schwinger principles, at least when applied to non-local quadratic actions, yield identical formulae. Via these formulae, we calculate the propagator associated with the action which is related to an electron gas in a random potential.

In earlier formulations of the problem of an electron gas in a random potential via Feynman's path integration (Bezák 1970, 1971), the following non-local quadratic action was considered:

$$S[\chi(t^*)] = \int_{t_0}^{t} dt^* \left((m/2)\dot{\chi}^2(t^*) - [m\Omega^2/4(t-t_0)] \int_{t_0}^{t} [\chi(t^*) - \chi(t')]^2 \, dt' \right). \tag{1}$$

Subsequently, this problem has been studied extensively by many other authors using different techniques (Dhara *et al* 1983, Khandekar *et al* 1981, 1983a, Makeshwari 1975, Papadopoulos 1974), as well as including more general forms of memory kernels in the second term of the integrand of (1) (Bosco 1983, Khandekar *et al* 1983b, 1986a, b, Zhang and Cheng 1986). The propagators resulting from all these calculations have been characterised by two distinct features: (i) they do not satisfy the composition property

$$K(x, t|x_0, t_0) = \int K(x, t|x', t') K(x', t'|x_0, t_0) \, dx' \tag{2}$$

and (ii) they are not of the Van Vleck-Pauli form

$$K_{VP}(x, t|x_0, t_0) = [(2\pi i \hbar)^{-1} |\partial^2 S_{cl}/\partial x \partial x_0|]^{1/2} \exp(iS_{cl}/\hbar). \tag{3}$$

The validity of the above-mentioned works has been disputed in Urrutia *et al* (1985), because these works do not give any reason by which their results satisfy neither (2) nor (3). While working in the framework of the Schwinger action principle, it has been hastily claimed by Urrutia *et al* (1985) that propagators related to non-local quadratic actions must fulfil (2) and (3). We show below, however, that this claim is unfounded. For this purpose, we provide physical and mathematical reasons by which propagators associated with non-local actions may not satisfy (2) and may not be of the form of (3). Furthermore, we demonstrate that the Feynman and Schwinger principles, at least when applied to non-local quadratic actions, may yield identical results.

Firstly, we begin by reasoning that the composition property (2) is strictly valid for propagators associated with additive actions with local Lagrangians. For non-local actions, which represent memory effects, the relation (2) does not hold anymore, even though the Feynman polygonal-path procedure is still meaningful, as has already been pointed out by Khandekar *et al* (1981) and references therein. To elucidate the last point, we notice that, for the action (1), $S(a, b)+S(b, c)\neq S(a, c)$. Only if the time intervals are equal, i.e. $(t_b-t_a)=(t_c-t_b)$, will the action in (1) be additive. However, the additive property is clearly satisfied by the Feynman polygonal-path procedure (corresponding to a discretisation of a path) where the total time interval is divided into *small* subintervals of equal length (Dhara *et al* 1983). In fact, when the fluctuation-dissipation theorem is taken into account, memory effects can well simulate dissipative mechanisms in quantum mechanics. The temporal reversibility of the resulting propagator is lost and clearly the composition property cannot prevail any longer. Results of this nature have often been encountered in a vast number of physical applications (Castigiano and Kokiatonis 1987, Grabert *et al* 1987, Hänggi 1986 and references therein, 1987, Leggett *et al* 1987 and references therein).

Secondly, one can readily verify that the Van Vleck–Pauli formula is not applicable for non-local quadratic actions (including (1)). Following Pauli (1981), we show that the Van Vleck–Pauli formula (3) does not satisfy the Schrödinger equation when the quadratic potential is non-local. Let us first write the Hamilton–Jacobi equation as

$$(\partial S/\partial t)+(1/2m)(\partial S/\partial x)^2+V(x, x_0)=0. \tag{4}$$

By differentiating (4) with respect to x and x_0 and defining $D\equiv(\partial^2 S/\partial x\partial x_0)$, one readily obtains that

$$D^{-1}(\partial D/\partial t)+m^{-1}(\partial^2 S/\partial x^2)+(mD)^{-1}(\partial D/\partial x)(\partial S/\partial x)+D^{-1}(\partial^2 V/\partial x\partial x_0)=0. \tag{5}$$

Now we form $(\mathrm{i}\hbar)(\partial K_{\mathrm{VP}}/\partial t)+(\hbar^2/2m)(\partial^2 K_{\mathrm{VP}}/\partial x^2)-VK_{\mathrm{VP}}$, where K_{VP} is given by (3). Then, by using (4) and (5), it follows that

$$\begin{aligned}(\mathrm{i}\hbar)(\partial K_{\mathrm{VP}}/\partial t)&+(\hbar^2/2m)(\partial^2 K_{\mathrm{VP}}/\partial x^2)-VK_{\mathrm{VP}}\\ &=-[(\hbar^2/2m\sqrt{D})(\partial^2\sqrt{D}/\partial x^2)+(\mathrm{i}\hbar/2D)(\partial^2 V/\partial x\partial x_0)]K_{\mathrm{VP}}.\end{aligned} \tag{6}$$

Even if D is independent of x (which is true when the potential is quadratic) K_{VP} is not yet the correct solution. Thus, the Van Vleck–Pauli formula is strictly valid only if the quadratic potential (and therefore the action) is local. This important point seems to have remained unnoticed in the literature.

As can be readily verified, the correct quantum mechanical Green function (or propagator) K for the Schrödinger equation (and valid for non-local quadratic actions) is given by (Nassar *et al* 1986)

$$K(x, t\,|\,x_0, t_0)=A\exp\left((\mathrm{i}S/\hbar)-\int_{t_0}^{t}(\mathrm{d}t^*/2m)(\partial^2 S/\partial x^2)\right) \tag{7}$$

where the constant of integration A is determined by the condition $\lim_{t\to t_0} K(x, t\,|\,x_0, t_0)=\delta(x-x_0)$.

Thirdly, we show next that the Schwinger action principle, at least when applied to non-local quadratic actions, yields an identical formula to (7), obtained previously in the framework of Feynman's principle (Nassar *et al* 1986).

Following Schwinger, we consider the infinitesimal variation in the transformation function $\langle x, t|x_0, t_0\rangle$ (Nassar and Machado 1987)

$$\delta\langle x, t|x_0, t_0\rangle = (\mathrm{i}/\hbar)\langle x, t|\delta W|x_0, t_0\rangle \tag{8}$$

where the set of quantum numbers x_0 and x, which label the initial and final states, are chosen at the times t_0 and t. $\delta W \equiv \delta W(x, t|x_0, t)$ is the infinitesimal Hermitian action operator assuming the representation $\delta W = \delta \int_{t_0}^{t} L\,\mathrm{d}t$, where L is the Hermitian Lagrangian operator. Let $\delta W = \delta \mathcal{W}$, where $\delta \mathcal{W}$ is the well ordered form of δW. The commutation properties of x and x_0 can be used to rearrange the operator W so that the x everywhere stand to the left of the x_0. Then

$$\delta\langle x, t|x_0, t_0\rangle = (\mathrm{i}/\hbar)\langle x, t|\delta\mathcal{W}|x_0, t_0\rangle = (\mathrm{i}/\hbar)\delta\mathcal{W}'\langle x, t|x_0, t_0\rangle \tag{9}$$

or

$$\langle x, t|x_0, t_0\rangle = \exp[(\mathrm{i}/\hbar)\mathcal{W}']. \tag{10}$$

This result resembles the customary Feynman path integral formula, but contains the eigenvalue of a well ordered exponent instead of a sum over paths (the primes on $\delta\mathcal{W}'$ and $\mathcal{W}'$ were simply used to make explicit the fact that these are eigenvalue quantities).

In turn, $\mathcal{W}$ satisfies the operator Hamilton–Jacobi (HJ) equation:

$$(\partial\mathcal{W}/\partial t) + (1/2m)(\partial\mathcal{W}/\partial x)^2 + V(x, x_0) = 0. \tag{11}$$

To solve it, we must bear in mind that in the limit $\hbar \to 0$ $\mathcal{W}$ becomes S, which satisfies the classical Hamilton–Jacobi equation

$$(\partial S/\partial t) + (1/2m)(\partial S/\partial x)^2 + V(x, x_0) = 0 \tag{12}$$

such that, to leading order of $\hbar$, we propose the ansatz

$$\mathcal{W} = \tilde{S} + \hbar\phi(t) \tag{13}$$

where $\tilde{S}$ means that the x are written to the left of the x_0 and the second term in (13) commutes with all operators and vanishes in the classical limit.

For a general time-dependent non-local quadratic system, the classical action can be formally written as

$$S = \tfrac{1}{2}[\alpha(t)x^2 + 2\beta(t)xx_0 + \gamma(t)x_0^2] + \sigma(t)x + \varepsilon(t)x_0 + \zeta(t). \tag{14}$$

(For convenience, we keep x and x_0 and only at the end do we set them to zero.) Accordingly, the momentum operator is

$$p = (\partial\mathcal{W}/\partial x) = (\partial\tilde{S}/\partial x) = \alpha x + \beta x_0 + \sigma \tag{15}$$

and the commutator of x and x_0 may be obtained from $[x, p] = \mathrm{i}\hbar$, which now becomes $\beta x_0 x = \beta x x_0 - \mathrm{i}\hbar$. Thus, we have the well ordered form

$$p^2 = (\alpha^2x^2 + \beta^2x_0^2 + 2\alpha\beta x x_0) + 2(\alpha x + \beta x_0)\sigma + \sigma^2 - \mathrm{i}\hbar\alpha \tag{16}$$

which, inserted together with (13) into the operator HJ (11) (and with the help of the corresponding classical HJ (12)), leaves us simply with

$$\dot{\phi} - (\mathrm{i}/2m)\alpha = 0. \tag{17}$$

Noting also that $\alpha = \partial^2 S/\partial x^2$, we obtain an *identical* expression to (7):

$$\langle x, t|x_0, t_0\rangle = A\exp\left((\mathrm{i}S/\hbar) - \int_{t_0}^{t}(\mathrm{d}t^*/2m)(\partial^2 S/\partial x^2)\right) \tag{18}$$

where the constant of integration A is determined by the condition $\lim_{t\to t_0}\langle x, t|x_0, t_0\rangle = \delta(x-x_0)$.

In particular, the non-local action (1) admits the classical equation

$$m\ddot{\chi} + m\Omega^2\chi = (m\Omega^2/t)\int_0^t \chi(t')\,\mathrm{d}t' \tag{19}$$

which can be solved with the boundary conditions $\chi(0) = x_0$ and $\chi(t) = x$ to yield the classical path

$$\chi(t^*) = \tfrac{1}{2}(x+x_0) + \{(x-x_0)\sin[(\Omega/2)(2t^*-t)]/2\sin(\Omega t/2)\}. \tag{20}$$

Now the action (1) as well as the propagator (7) (or (18)) can be readily evaluated:

$$S_{\mathrm{cl}} = [(m\Omega/4)\cot(\Omega t/2)(x-x_0)^2] \tag{21}$$

$$K = [\cos(\Omega t/2)]^{-1/2}[(m\Omega/4\pi\mathrm{i}\hbar)\cot(\Omega t/2)]^{1/2}\exp[\mathrm{i}S_{\mathrm{cl}}/\hbar]. \tag{22}$$

This propagator does not satisfy the composition property (2) and is not of the form of the Van Vleck-Pauli formula (3) either, for the reasons discussed above.

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